CS202 Midterm Exam

October 24th, 2007

Write your answers on the exam. Justify your answers. Work alone. Do not use any notes or books.

There are four problems on this exam, each worth 20 points, for a total of 80 points. You have approximately 50 minutes to complete this exam.

1 Dueling recurrences (20 points)

Let $0 \leq S(0) \leq T(0)$, and suppose we have the recurrences

$$S(n+1) = aS(n) + f(n)$$
$$T(n+1) = bT(n) + g(n),$$

where $0 \leq a \leq b$ and $0 \leq f(n) \leq g(n)$ for all $n \in \mathbb{N}$.

Prove that $S(n) \leq T(n)$ for all $n \in \mathbb{N}$.

2 Seating arrangements (20 points)

A group of k students sit in a row of n seats. The students can choose whatever seats they wish, provided: (a) from left to right, they are seated in alphabetical order; and (b) each student has an empty seat immediately to his or her right.

For example, with 3 students `A`, `B`, and `C` and 7 seats, there are exactly 4 ways to seat the students: `A-B-C--`, `A-B--C-`, `A--B-C-`, and `-A-B-C-`.

Give a formula that gives the number of ways to seat k students in n seats according to the rules given above.

3 Non-attacking rooks (20 points)

Place n rooks at random on an $n \times n$ chessboard (i.e., an $n \times n$ grid), so that all $\binom{n^2}{n}$ placements are equally likely. What is the probability of the event that every row and every column of the chessboard contains exactly one rook?

4 Subsets (20 points)

Let $A \subseteq B$.

1. Prove or disprove: There exists an injection $f : A \to B$.
2. Prove or disprove: There exists a surjection $g : B \to A$.